ON QUALITATIVE SINGULARITIES OF THE FLOW OF A VISCOPLASTIC MEDIUM IN PIPES

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The present paper directly extends from [1 and 2], and is devoted to the study of qualitative singularities of the flow of a viscoplastic medium in cylindrical pipes with arbitrary cross-section under the effect of a constant pressure drop at the ends of the pipe.

The method of qualitative analysis of the motion utilized herein permits obtaining a global condition for the existence of stagnant zones, which is connected with the configuration of the pipe cross-section, and refinement of the location of the dead zones in a number of cases.

Moreover, a more accurate estimate for the size of the flow nuclei is obtained. Namely, in the case of a simply connected cross-section each nucleus contains a circle of radius greater than τ_0/c, where the width of the flow nucleus is exactly equal to τ_0/c for the case of flow in a plane-parallel gap.

The results obtained rely on the following lemma.

Lemma. If a continuously differentiable function $u(x, y)$ is defined in an open doubly connected domain ω with inner boundary Γ_1 and outer boundary Γ_0, when $u|_{\Gamma_0} = 0$, $u|_{\Gamma_1} = 1$, then

$$\int_\omega |\nabla u| \, d\omega \geqslant \inf_\Gamma \operatorname{mes} \Gamma \tag{1}$$

where the right-hand side of (1) is the lower bound of the lengths of closed curves Γ in ω, which are homotopic to the boundaries Γ_0, and Γ_1.(*)

Proof. Evidently it is sufficient to establish the inequality (1) on continuously differentiable functions in ω, which have no local maxima and minima. It follows from the Sard theorem [3] that a set of measure zero, of values of the function $u(x, y)$ may be removed such that the remaining values $u = \rho$ correspond to level lines Γ_ρ, in whose neighborhood $O(\Gamma_\rho)$ local coordinates may be introduced by selecting the arclength along Γ_ρ as one coordinate s, and the normal to Γ_ρ as the other n (the positive direction of the normal corresponds to growth of the function $u(x, y)$).

Let us consider the domain $O_1 = O(\rho, \rho + \Delta\rho) \subset O(\Gamma_\rho)$, included between the level lines Γ_ρ, $\Gamma_{\rho+\Delta\rho}$, such that

$$dx\,dy = I(s, n)\, ds\,dn, \quad |I(s, n) - 1| < \varepsilon \quad \text{в } O_1$$

Then

$$\int_{O_1} |\nabla u| \, d\omega \geqslant (1 - \varepsilon)\, \Delta\rho \operatorname{mes} \Gamma_\rho \tag{2}$$

Evidently, for any $\varepsilon > 0$ a finite sequence $\rho_i (i = 0, 1, \ldots, 2n(\varepsilon))$ may be selected such that ρ_i is a noncritical value for $u(x, y)$ in the domain

$$O_i = O(\rho_{2i}, \rho_{2i} + \Delta\rho_{2i}), \qquad \Delta\rho_{2i} = \rho_{2i+1} - \rho_{2i}$$

the inequality (2) is satisfied, and

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*) We shall designate two closed continuous curves Γ_1, Γ_2, lying in ω, as homotopic to each other if one contour can go over into the other by a continuous deformation in the domain ω.

$$\sum_{0}^{n(\varepsilon)} \Delta\rho_{2i} > 1 - \varepsilon$$

Evidently

$$\int_{\omega} |\nabla u|\, d\omega \geqslant \sum_{0}^{n(\varepsilon)} \int_{O_i} |\nabla u|\, d\omega \geqslant (1-\varepsilon) \sum_{0}^{n(\varepsilon)} \Delta\rho_{2i} \operatorname{mes} \Gamma_{\rho_{2i}} \tag{3}$$

Let us consider γ(*) homotopic to Γ_ρ in ω, which has least length among all closed curves homotopic to Γ_ρ in ω. Evidently the contour γ coincides with Γ_1 if Γ_1 is convex; if Γ_0 is not convex, then γ coincides with the convex shell Γ_1. In the general case, the contour γ is a segment tangent to the curves Γ_1, Γ_0 (Fig. 1) in the neighborhood of each point of ω.

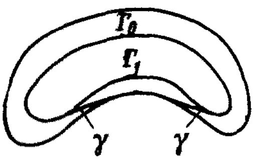

Fig. 1

It follows from the inequality (3) that

$$\int_{\omega} |\nabla u|\, d\omega \geqslant (1-\varepsilon)^2 \operatorname{mes} \gamma \tag{4}$$

Since ε is arbitrary, the inequality (4) is equivalent to the inequality (1).

Let us note that the inequality (1) is exact. Indeed, it is possible to construct a sequence of continuously differentiable functions $u_n(x, y)$, $u_n|_{\Gamma_0} = 0$, $u_n|_{\Gamma_1} = 1$, which converges to the characteristic function of the domain bounded by the contour γ, and such that

$$\lim_{n\to\infty} \int_{\omega} |\nabla u_n|\, d\omega = \operatorname{mes} \gamma$$

Let us consider the problem of stationary motion of a viscoplastic medium in a cylindrical pipe under the effect of a pressure gradient. Let ω denote the pipe cross-section. Let Γ be the boundary of ω. For simplicity, let us assume that ω is a simply-connected domain. Stationary motion is characterized by the velocity distribution $u(x, y)$ in the domain ω. According to [1], the true motion is separated out of all those kinematically possible by the condition that the functional

$$J(v) = \int_{\omega} \left\{ \frac{\mu}{2} |\nabla u|^2 + \tau_0 |\nabla v| - cv \right\} d\omega, \qquad v|_\Gamma = 0 \tag{5}$$

reaches its minimum. All velocity distributions $v(x, y)$ belonging to the space $W_2^{(1)}(\omega)$ (see [4]), and satisfying the condition $v|_\Gamma = 0$ are kinematically possible. The existence and uniqueness of the velocity distribution $u(x, y)$ minimizing (5), were proved in [1]. This distribution $u(x, y)$ belongs to the space $W_2^{(1)}(\omega)$ and is a continuous function, where $u(x, y)$ is positive, has no local minima, and a finite number of local maxima.

Let us consider the open set ω_ρ consisting of points of the domain ω such that $u(x, y) > \rho$. The set ω_ρ is the union of a finite number of open simply connected domains ω_ρ^ν; $\omega_\rho = \bigcup_1^{n(\rho)} \omega_\rho^\nu$. There evidently exists a sufficiently small number $\Delta\rho$ such that

$$n(\rho) = n(\rho + \Delta\rho), \quad \omega_{\rho+\Delta\rho} = \bigcup_1^{n(\rho)} \omega_{\rho+\Delta\rho}^\nu, \quad \omega_{\rho+\Delta\rho}^\nu \subset \omega_\rho^\nu$$

We shall designate the boundary Γ_ρ of the domain ω_ρ the level line of the function $u(x, y)$. Evidently $\Gamma_\rho = \bigcup_1^{n(\rho)} \Gamma_\rho^\nu$. Let $\omega_{\rho,\rho+\Delta\rho}^\nu$ denote the set $\omega_{\rho,\rho+\Delta\rho}^\nu = \omega_\rho^\nu / \overline{\omega}_{\rho+\Delta\rho}^\nu$.

Theorem 1. The level lines Γ_ρ of the minimizing function are finite

$$\operatorname{mes} \Gamma_\rho^\nu \leqslant \frac{c}{\tau_0} \operatorname{mes} \omega_\rho^\nu \tag{6}$$

Proof. Let us consider the function $u^*(x, y)$

$$u^*(x, y) = u(x, y), \qquad \text{если } (x, y) \overline{\in} \omega_\rho^\nu$$

$$u^*(x, y) = \rho, \qquad \text{если } (x, y) \in \omega_{\rho,\rho+\Delta\rho}^\nu$$

$$u^*(x, y) = u(x, y) - \Delta\rho, \quad \text{если } (x, y) \in \overline{\omega}_{\rho+\Delta\rho}^\nu$$

Evidently $J(u^*) > J(u)$, from which it directly follows that

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*) A closed curve which is mutually one-to-one and mutually continuous image of a circle is called a contour.

$$\tau_0 \int\limits_{\omega^{\nu}_{\rho,\rho+\Delta\rho}} |\nabla u|\, d\omega \leqslant c\Delta\rho \operatorname{mes} \omega_\rho^{\nu}$$

Utilizing Lemma, we find

$$\operatorname{mes} \gamma^{\nu}_{\rho,\rho+\Delta\rho} \leqslant \frac{c}{\tau_0} \operatorname{mes} \omega_\rho^{\nu} \tag{7}$$

Here $\gamma^{\nu}_{\rho,\rho+\Delta\rho}$ is the contour introduced in Lemma, which lies in the domain $\omega^{\nu}_{\rho,\rho+\Delta\rho}$. Furthermore, $\omega^{\nu}_{\rho+\Delta\rho_1} \supset \omega^{\nu}_{\rho+\Delta\rho_2}$, if $\Delta\rho_1 < \Delta\rho_2$ and $\lim \omega^{\nu}_{\rho+\Delta\rho} = \omega_\rho^{\nu}$ as $\Delta\rho \to 0$. It hence follows that if Γ_ρ^{ν} is infinite, then $\lim \operatorname{mes} \gamma^{\nu}_{\rho,\rho+\Delta\rho} = \infty$ as $\Delta\rho \to 0$, which contradicts the inequality (7). Therefore, inequality (6) is satisfied and Γ_ρ^{ν} is finite. The theorem is proved.

Theorem 2. If the boundary Γ of the domain ω and the number c in the functional (5) are such that

$$\tau_0 \left(\frac{\operatorname{mes}\omega}{\operatorname{mes}\Gamma}\right)^{-1} > c > \tau_0 \left(\sup_{\omega' \subseteq \omega} \frac{\operatorname{mes}\omega'}{\operatorname{mes}\Gamma'}\right)^{-1}$$

where Γ' is the boundary of a subdomain ω', then the flow in the domain ω exists and has stagnant zones, i.e., domains adjoining the boundary Γ, where $u \equiv 0$.

Proof. The existence of the flow follows from the necessary and sufficient condition formulated in [1].

Let us prove the existence of the stagnant zones. Let us assume the opposite, then Γ is the level line $u = 0$, and it follows from Theorem 1 that $\operatorname{mes}\Gamma \leqslant (c/\tau_0) \operatorname{mes}\omega$, which contradicts the assumption of Theorem 2.

The physical meaning of the assertion in Theorem 2 is quite simple. In fact, Theorem 2 permits the conclusion that for some sufficiently small c the boundary of the domain ω can not be a line bounding the flow domain, which always holds when the contact contour does not coincide with the boundary of the domain ω. The case of flow in a pipe with square cross-section [1] might be mentioned as a specific example.

Theorem 3. The flow nucleus A (*) with boundary a is such that

$$\operatorname{mes} a \leqslant \frac{c}{\tau_0} \operatorname{mes} A \tag{8}$$

and contains a circle of radius

$$R \geqslant \frac{2\tau_0}{c}\left[1 + \left(1 - \frac{4\pi\tau_0^2}{c^2 \operatorname{mes}\omega}\right)^{1/2}\right]^{-1}$$

Proof. The inequality (8) follows directly from Theorem 1. Let us estimate the magnitude of R. Burago and Zalgaller [5] established an inequality which has the form

$$\operatorname{mes} A \leqslant R \operatorname{mes} a - \pi R^2$$

when A is a plane simply connected domain with boundary a and internal radius R (**).

Hence, utilizing the inequality (9), we find

$$R \geqslant \frac{2\operatorname{mes} A}{\operatorname{mes} a + \sqrt{(\operatorname{mes} a)^2 - 4\pi \operatorname{mes} A}} \geqslant \frac{2\tau_0}{c}\left[1 + \left(1 - \frac{4\pi\tau_0^2}{c^2 \operatorname{mes}\omega}\right)^{1/2}\right]^{-1} > \frac{\tau_0}{c}$$

Let us note that upper and lower estimates for R were obtained in [1]. The upper estimate $R \leqslant 2\tau_0/c$ was exact, the lower estimates were rough. The lower estimate (9) is apparently almost exact since $R = \tau_0/c$ if ω is a strip; let us note that the radicand in (9) is positive when a flow exists. The theorem is proved. The expounded methods are also applicable if the flow nuclei have a multiconnected configuration; however, the estimates obtained here are essentially rougher.

Theorem 1 also permits mention of the following property of the nucleus. For each nucleus A of boundary a

$$\tau_0 \operatorname{mes} a = c \operatorname{mes} A$$

and for any subdomain A' in A with boundary a'

$$\tau_0 \operatorname{mes} a' \geqslant c \operatorname{mes} A'$$

The last inequality shows, for example, that the nucleus can not have angular points directed toward the flow. Such an impossible nucleus configuration is shown in Fig. 2.

Let us note yet another qualitative peculiarity of stagnant zones. The closed curves K_α

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*) An inner subdomain of ω, in which $u(x, y)$ is constant and achieves its local maximum is called the flow nucleus (see [2]).

**) A number R such that $R = \sup \rho(x, a)$, where $\rho(x, a)$ is the distance from a point x of the domain A to the boundary a, is called the internal radius of the domain A.

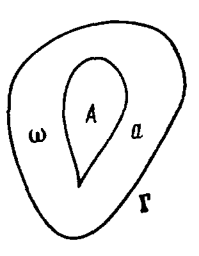

Fig. 2

defined by the condition (mes Ω_a/mes K_a) = sup (mes ω'/mes Γ'), where Ω_a is a domain in ω, bounded by the curve K_a; Γ' is the boundary of ω', were introduced in [1]. Let us assume that there exists just one closed curve K, which yields the upper bound of the ratio mes ω'/mes Γ'; then if the motion exists, the stagnant zone is outside the domain Ω, bounded by the curve K. Let us first prove the following assertion.

Theorem 4. Let

$$J_i(u)=\int_\omega\left\{\frac{\mu}{2}|\nabla u|^2+\tau_0|\nabla u|-c_iu\right\}d\omega,\qquad u|_\Gamma=0,\qquad i=1,\,2$$

If u_i is a function minimizing $J_i(u)$ and $c_1\geqslant c_2$, then $u_1\geqslant u_2$, i.e. as the pressure gradient increases, the velocity of the motion can not decrease at any point of the domain ω.

Proof. Evidently the chain of inequalities

$$J_1(u_1)\leqslant J_1(u_2)\leqslant J_2(u_2)\leqslant J_2(u_1)$$

holds.

Let us assume the opposite. Let $u_2>u_1$ in the domain ω^* and $u_2=u_1$ on Γ^*, Γ^* is the boundary of ω^*. Evidently an analogous chain of inequalities holds

$$J_1^*(u_1)\leqslant J_1^*(u_2)\leqslant J_2^*(u_2)\leqslant J_2^*(u_1)$$

$$J_i^*(u)=\int_{\omega^*}\left\{\frac{\mu}{2}|\nabla u|^2+\tau_0|\nabla u|-c_iu\right\}d\omega,\qquad i=1,\,2$$

But then

$$\int_{\omega^*}\left\{\frac{\mu}{2}|\nabla u_1|^2+\tau_0|\nabla u_1|\right\}d\omega\leqslant\int_{\omega^*}\left\{\frac{\mu}{2}|\nabla u_2|^2+\tau_0|\nabla u_2|-c_1(u_2-u_1)\right\}d\omega$$

$$\int_{\omega^*}\left\{\frac{\mu}{2}|\nabla u_2|^2+\tau_0|\nabla u_2|\right\}d\omega\leqslant\int_{\omega^*}\left\{\frac{\mu}{2}|\nabla u_1|^2+\tau_0|\nabla u_1|-c_2(u_1-u_2)\right\}d\omega$$

Combining the last two inequalities we find

$$c_1A\leqslant c_2A,\qquad A=\int_{\omega^*}(u_2-u_1)\,d\omega>0$$

Since $A>0$, then $c_1\leqslant c_2$, which contradicts the condition of the theorem.

Theorem 5. If just one closed curve K bounding the domain Ω and yielding the upper bound of the ratio mes ω'/mes Γ' exists, where ω' is a subdomain of ω with boundary Γ', and if the motion of the medium exists in ω then the stagnant zones of the motion of the viscoplastic medium are outside the domain Ω.

Proof. Let us consider the sequence of functionals

$$J_i(u)=\int_\omega\left\{\frac{\mu_i}{2}|\nabla u|^2+\tau_0|\nabla u|-c_iu\right\}d\omega,\qquad |u|_\Gamma=0$$

Here the numbers c_i, by decreasing, tend to the number $c^*=\tau_0$ mes K/mes Ω, the numbers $\mu_i\to 0$, such that max $u_i=1$, where u_i is the function minimizing $J_i(u)$. It is easy to see that $J(u_i)\to 0$, $i\to\infty$ and therefore

$$\int_\omega\frac{\mu_i}{2}|\nabla u_i|^2\,d\omega\to 0,\qquad 0\leqslant\int_\omega\{\tau_0|\nabla u_i|-c^*u_i\}\,d\omega\to 0\qquad(i\to\infty)$$

Let us consider the subdomains $\omega_\rho^{\,i}$ in ω (see Theorem 1), which are representable as

$$\omega_\rho^{\,i}=\bigcup\nolimits_1^{n(\rho,\,i)}\omega_\rho^{\,i,\,\nu}$$

Let $\Gamma_\rho^{i,\nu}$ denote the boundary of $\omega_\rho^{i,\nu}$. It follows from Theorem 1 that

$$\sum_\rho \text{mes}\,\Gamma_\rho^{\,i,\,\nu}\leqslant\frac{c^*}{\tau_0}\sum_\rho\text{mes}\,\omega_\rho^{\,i,\,\nu}+\varepsilon_i \tag{9}$$

Here $\varepsilon_i\to 0$ as $i\to\infty$.

Let us rewrite inequality (9) as

$$\sum_\rho\text{mes}\,\Gamma_\rho^{\,i,\,\nu}\leqslant\sum_\rho\text{mes}\,\Gamma_\rho^{\,i,\,\nu}\left[\frac{\text{mes}\,\omega_\rho^{\,i,\,\nu}}{\text{mes}\,\Gamma_\rho^{\,i,\,\nu}}\left(\sup\frac{\text{mes}\,\omega'}{\text{mes}\,\Gamma'}\right)^{-1}\right]+\varepsilon_i \tag{10}$$

From the inequality (10) it follows that

$$\int_{\omega} | u_i - \theta(x, y) | \, d\omega \to 0 \qquad (11)$$

Here $\theta(x, y)$ is the characteristic function of Ω. Let us assume that a certain function $u_i(x, y)$ has a stagnant zone intersecting the domain Ω. It follows from Theorem 4 that the functions $u_k(x, y)$, for $k > i$, have stagnant zones containing the stagnant zone of the function $u_i(x, y)$. Therefore, (11) is impossible. The obtained contradiction proved Theorem 5.

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